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SOME NOTES ON THE DEVELOPMENT OF THE HYDRODYNAMIC THEORY OF BOILING

by

Vijay Dhir

(based on 3 lectures by J. H. Lienhard)

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Lexington, Kentucky

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PREFACE

In January 1970, the personnel of the Gravity Boiling Project included six new people, all generally unfamiliar with Zuber's hydrodynamic theory of the extreme boiling heat fluxes. Since a great deal of the work we do depends upon the concepts that make up this theory, some remedial work was required. We therefore agreed to meet for three one and one-half hour informal lectures, first to review some of the underlying wave theory, and then to trace Zuber's predictions.

Vijay Dhir then undertook to put these talks into order, filling in the missing words and equations, correcting errors, and improving some of the logic. We hope that the result will serve in the future as a kind of instructional package for other people who wish to begin work with the extreme pool boiling heat fluxes.

The first Section of the report (pages 1 through 19) provides a brief background in the Taylor and Helmholtz stability of waves. This discussion is slanted toward an application to the hydrodynamic theory. Nevertheless, the reader with a background in this material should comfortably be able to begin directly in Section II.

J. H. Lienhard

I. DYNAMICS AND STABILITY OF SMALL GRAVITY AND CAPILLARY WAVES*

A. Gravity Waves

1. Wave Motion

A wave motion of a liquid acted upon by gravity and having a free surface is a motion in which the elevation of the free surface above some chosen fixed horizontal plane varies. We shall restrict our remarks to two-dimensional motion for the present.

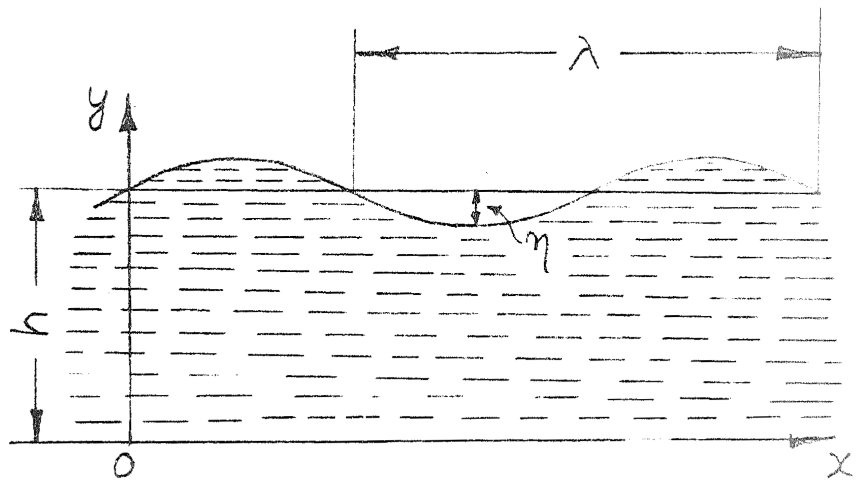


Fig. 1 Wave Motion on the Free Surface of a Liquid

With reference to Fig. 1 we shall take the axis of x to be horizontal and the axis of y to be vertically upwards; the motion of the free surface governed by equation (1)

$$\eta = a \sin(kx - \omega t) \tag{1}$$

is called a simple harmonic progressive wave where $k = 2\pi/\lambda$,

*A good general reference is L. M. Milne-Thomson, Theoretical Hydrodynamics, [1].

the wave number, and ω is the frequency.

2. Kinematical Condition at the Free Surface

Considering a fluid of depth h in which waves of height $\eta = \eta(x,t)$ above the mean level are propagated, the equation of the free surface is

$$y = \eta + h \tag{2}$$

As the surface moves with the fluid, following a fluid particle we can write

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} = v \tag{3}$$

Assuming a linearised theory in which squares and products of variable parts of all quantities and their differential coefficients are negligible, and, in our case taking the slope, $\frac{\partial \eta}{\partial x}$, of the profile to be small, we get

$$\frac{\partial \eta}{\partial t} = v \tag{4}$$

Assuming the motion to be irrotational we can define a potential function, ϕ, and a stream function, ψ, so that

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} \tag{5}$$

Equation (5) is kinematical surface condition for wave profiles of small height and slope.

Now let us choose a complex potential of the form

$$W = \phi + i\psi = A \cos(kz - \omega t)$$
$$= A\{\cos(kx-\omega t)\cosh ky - i[\sin(kx-\omega t)\sinh ky]\}$$

and determine what A must be to fit the boundaries. The potential and stream functions for this potential are

$$\phi = A \cos(kx - \omega t) \cosh ky$$

$$\psi = -A \sin(kx - \omega t) \sinh ky$$

At the surface ψ becomes

$$\psi_{surface} = -A \sin(kx - \omega t) \sinh kh$$

where we have assumed $y = h + \eta \simeq h$ since η is small. However

$$\eta = a \sin(kx - \omega t) \tag{1}$$

Substituting equation (1) in (5) we get:

$$A = \frac{a\omega/k}{\sinh kh}$$

Thus

$$\omega = \frac{ac}{\sinh kh} \cos(kz - \omega t) \tag{6}$$

We have derived this result from purely kinematic conditions and no hypothesis has been made as to conditions above the wave profile.

3. Pressure Condition at the Free Surface

Let p_i be the pressure inside the liquid and p_o, the pressure just outside. Assuming the fluid to be inviscid, we can write the equation of motion as

$$\rho \frac{D\vec{u}}{Dt} = \rho\vec{g} - \vec{\nabla}p_i$$

or

$$\rho \frac{\partial \vec{u}}{\partial t} + \vec{\nabla}(\tfrac{1}{2}\vec{u}^2\rho) = -\vec{\nabla}(p_i - \rho g\eta)$$

or

$$\vec{\nabla}(\rho \frac{\partial\phi}{\partial t} - \rho g\eta + \frac{\rho}{2}\vec{u}^2 + p_i) = 0$$

Integrating this with respect to position we get

$$\rho \frac{\partial\phi}{\partial t} - \rho g\eta + \frac{\rho}{2}\vec{u}^2 + p_i = c(t).$$

$c(t)$ can be taken to be independent of t by incorporating any time variable in $\frac{\partial\phi}{\partial t}$. Also if p_o be the constant outside pressure acting at the interface, we can absorb it in ϕ without any loss of generality and finally write

$$p_i - p_o = \rho(\frac{\partial\phi}{\partial t} - g\eta + \frac{\vec{u}^2}{2}) \qquad (7)$$

Neglecting $\frac{\vec{u}^2}{2}$ which is negligible in small waves, we obtain

$$p_i - p_o = \rho(\frac{\partial\phi}{\partial t} - g\eta) \qquad (8)$$

The interface between two fluids which do not mix behaves as if it were in a state of uniform tension. This tension--called surface tension--depends on the nature of the two fluids and on temperature. We now wish to see what effect this tension has on p. Figure 2 shows the surface tension and pressure forces acting on an element δS of arc of a cross-section of a cylindrical surface forming the interface between two fluids. If $\delta\theta$ is the angle between the tangents at P and S, then resolving the forces along the normal at P gives approximately

$$-p_o\delta S + p_i\delta S + \sigma\delta\theta = 0$$

or

$$p_i - p_o = - \frac{\sigma \delta \theta}{\delta S}$$

or

$$p_o - p_i = \frac{\sigma}{R_{xy}}$$

where R_{xy} is the radius of curvature. Using the expression for

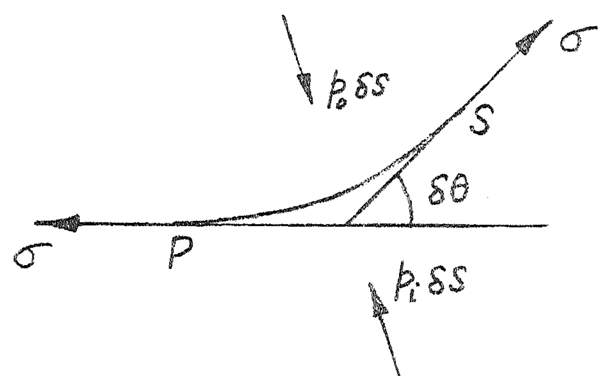


Fig. 2 Forces Acting on the Interface between Two Immiscible Fluids

R_{xy} in terms of η,

$$p_o - p_i = \frac{\sigma \dfrac{\partial^2 \eta}{\partial x^2}}{\left(1 + \left(\dfrac{\partial \eta}{\partial x}\right)^2\right)^{3/2}} \approx \sigma \frac{\partial^2 \eta}{\partial x^2} \tag{9}$$

since

$$\frac{\partial \eta}{\partial x} << 1$$

Substituting equation (9) in (8) we get

$$-\sigma \frac{\partial^2 \eta}{\partial x^2} = \rho \left(\frac{\partial \phi}{\partial t} - g\eta\right) \tag{10}$$

In the simplest case in which we neglect the surface tension, we get at the free surface

$$\frac{\partial \phi}{\partial t} - g\eta = 0 \tag{11}$$

Notice that this equation is independent of fluid density and gives surface elevation when ϕ is known.

At the free surface

$$\phi = \phi(x,h+\eta,t) = \phi(x,h,t) + \eta\left(\frac{\partial \phi(x,y,t)}{\partial y}\right)_{y=h} + \ldots$$

As η is small, the second term on right hand side can also be neglected and we get at the surface

$$\phi \simeq \phi(x,h,t)$$

Therefore equation (11) becomes at the surface

$$g\eta = \left.\frac{\partial \phi}{\partial t}\right|_{y=h}$$

or

$$\left.\frac{\partial \phi}{\partial t}\right|_{y=h} - g\eta = 0 \tag{12}$$

and

$$\left.\frac{\partial^2 \phi}{\partial t^2}\right|_{y=h} - g\frac{\partial \eta}{\partial t} = 0 \tag{13}$$

4. Surface Waves

Combining the kinematical and pressure conditions, we get

$$\left.\frac{\partial^2 \phi}{\partial t^2}\right|_{y=h} - g\frac{\partial \psi}{\partial x} = 0 \tag{14}$$

Substituting the known values of ϕ and ψ in equation (14) we get the wave speed, c^2;

$$c^2 = \frac{g}{k} \tanh kh$$

Now taking special cases

(i) Waves on a very deep fluid ($h>>\lambda$):

In the Limit as $kh\to\infty$, $\tanh kh \to 1$, therefore

$$c^2 = \frac{g}{k}$$

or

$$c = \sqrt{\frac{g\lambda}{2\pi}}$$

(ii) Waves on a very shallow fluid ($kh<<1$):

In the limit, as kh becomes small

$$c^2 = \frac{g}{k}\,\frac{1+kh-1+kh}{1+kh+1-kh} = gh$$

or

$$c = \sqrt{gh}$$

<u>Example</u> Tidal waves at sea (shallow waves)

Typically: $a = 3$ ft

$\omega = 4\pi$ rad/hr

$\lambda = 200$ miles

$h = 12{,}000$ ft

Therefore,

$$c = \sqrt{32.2\text{x}12{,}000}$$

$$= 622 \text{ ft/sec}$$

Thus for very deep liquid ($\lambda/h<<1$) c is proportional to $\sqrt{\lambda}$ and for shallow liquid ($\lambda/h>>1$) $c = \sqrt{gh}$, <u>i.e.</u> c tends to a constant value which it cannot exceed. The results are shown in Figure 3.

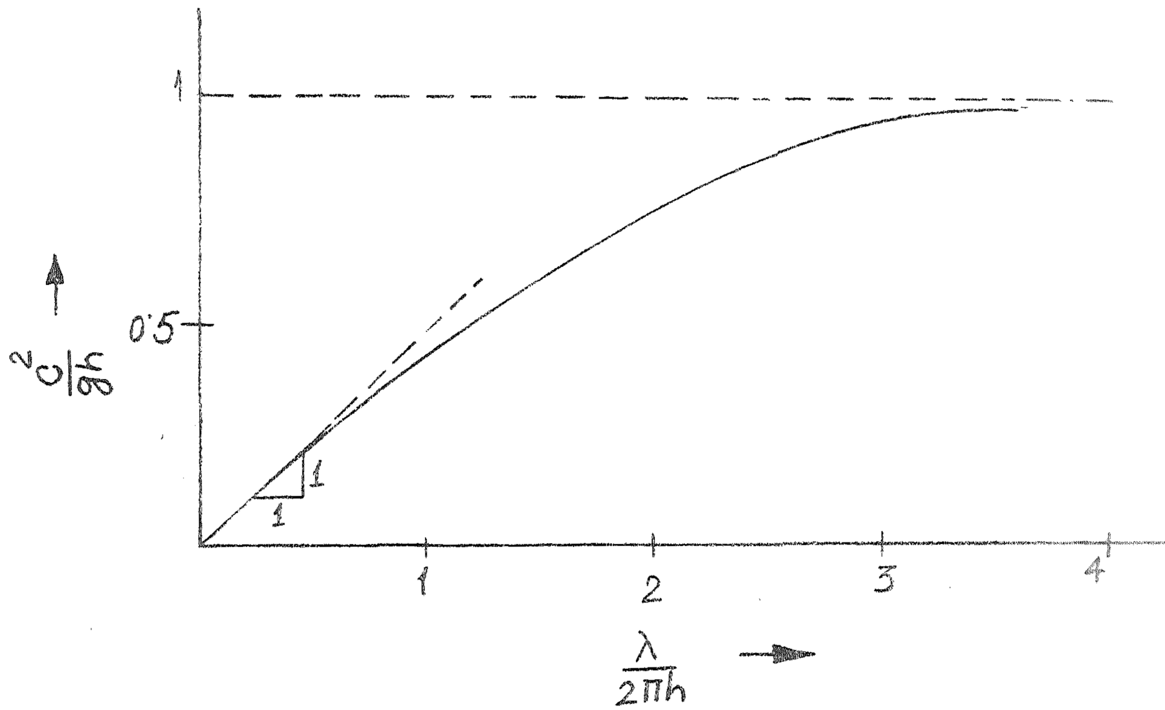


Fig. 3 Effect of Depth of Liquid on Wave Velocity

5. Gravity Waves in Two Fluids (Waves at an Interface)

Consider a fluid of density, ρ', and depth, h', flowing with constant velocity, U', over a layer of fluid of density, ρ, and depth, h, which flows with constant velocity, U, the fluids being bounded above and below by rigid horizontal planes as shown in Figure 4.

To study the situation that a wave of small elevation $\eta = a \sin(kx - \omega t)$ may be propagated at the interface with velocity, c, we impose on the whole mass of fluid a velocity, c, opposite to the direction of propagation; thus reducing the profile to rest and changing the velocities of streams to $U'-c$ and $U-c$.

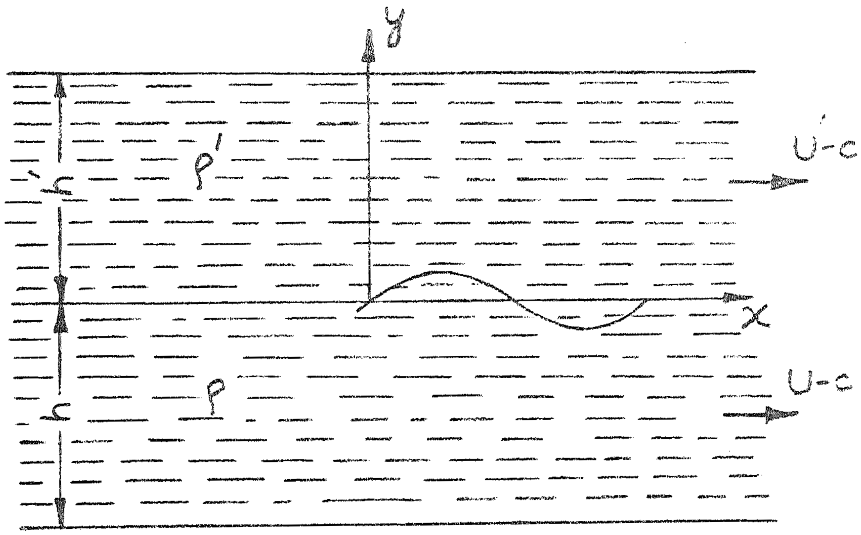


Fig. 4 Interface between Two Different Fluids Flowing with Different Velocities

The complex potential for the lower fluid is

$$W = -(U-c)z - \frac{a(U-c)}{\sinh kh} \cos k(z+ih) \tag{15}$$

This complex potential satisfies the condition that streamline $\psi=0$ corresponds to $\eta = a \sin kx$. Also the potential function, ϕ, and stream function, ψ, satisfy Laplace's Equation.

Similarly we write the complex potential for the upper fluid by replacing h with -h':

$$W' = -(U'-c)z + \frac{a(U'-c)}{\sinh kh'} \cos k(z-ih') \tag{16}$$

Therefore

$$\phi = -(U-c)x - \frac{a(U-c)}{\sinh kh} \cos kx \cosh k(y+h)$$

$$\phi' = -(U-c)x + \frac{a(U'-c)}{\sinh kh'} \cos kx \cosh k(y-h')$$

or

$$u = \frac{\partial \phi}{\partial x} = -(U-c) + \frac{ka(U-c)}{\sinh kh} \sin kx \cosh k(y+h)$$

$$u' = \frac{\partial \phi'}{\partial x} = -(U'-c) - \frac{ka(U'-c)}{\sinh kh'} \sin kx \cosh k(y-h')$$

Neglecting terms containing a^2 and applying the condition that at the interface, $y = \eta$ and $\eta << h$, we get at the interface

$$u^2 = (U-c)^2 \; [1-2k\eta \coth kh] \tag{17}$$

$$u'^2 = (U'-c)^2 \; [1+2k\eta \coth kh'] \tag{18}$$

Now applying Bernoulli's Equation at the interface for the two fluids we have

$$p' + \frac{1}{2} \rho' u'^2 + \rho' g\eta = \text{constant} \tag{19}$$

and

$$p + \frac{1}{2} \rho u^2 + \rho g\eta = \text{constant} \tag{20}$$

But the pressure must be continuous at the interface (assuming no surface tension), so

$$p = p'$$

Subtracting equation (20) from (19); substituting values of u^2 and u'^2 from (17) and (18) and assuming that free stream kinetic energy per unit volume for the two streams is the same, we get the condition

$$\rho' k (U'-c)^2 \coth kh' + \rho k (U-c)^2 \coth kh = g(\rho-\rho')$$

If $U = U' = 0$ [both fluids stationary], we get

$$c^2 = \frac{g(\rho-\rho')}{k(\rho \coth kh + \rho' \coth kh')} \tag{21}$$

If the depth is great, <u>i.e.</u> kh and $kh' >> 1$, we get

$$c^2 = \frac{g(\rho-\rho')}{k(\rho+\rho')} \tag{22}$$

The condition for stability of waves is that c should be real. Thus ρ must be greater than ρ', *i.e.* the heavier fluid must be underneath.

B. Capillary-Gravity Waves

We proceed with the assumption that fluid depths are large and the waves are unsteady and of small amplitude.

We define:

$$\Phi \equiv Ux + \phi \qquad \text{and} \qquad p_t \equiv p_o + p$$

where

ϕ = perturbation potential function

U = free stream velocity

p_o = static pressure

p = perturbation or disturbance pressure

As the governing equations are linear under small wave assumptions, we can treat one component of the Fourier spectrum, throwing away others, and write for ka small

$$\eta = a \exp[i(\omega t - kx)] \tag{23}$$

The complex potentials that match this form of η are

$$\phi = i(U-c)a \exp[ky + i(\omega t - kx)] \tag{24}$$

$$\phi' = -i(U'-c)a \exp[-ky + i(\omega t - kx)] \tag{25}$$

These potential functions are correct at infinity and at the interface, and they satisfy Laplace's Equation.

Writing the pressure equation

$$-g\eta + \frac{\partial \Phi}{\partial t} + \frac{1}{2}\vec{u}^2 = \frac{p_t}{\rho} \tag{26}$$

and defining $\vec{u} = U + \vec{\mu} = \vec{\nabla}\Phi$

$$\vec{\mu} = \vec{\nabla}\phi$$

Neglecting terms on the order of μ^2 we rewrite the pressure equation as

$$-g\eta + \frac{\partial \phi}{\partial t} + \frac{1}{2}U^2 + \mu U = \frac{p_o}{\rho} + \frac{p}{\rho}$$

But in the free stream $\frac{p_o}{\rho} = \frac{1}{2}U^2 + \text{constant.}$

Hence the pressure equation for the lower fluid becomes

$$\rho[-g\eta + \phi_t + \phi_x U] = p \tag{27}$$

Similarly for the upper fluid

$$\rho'[-g\eta + \phi'_t + \phi'_x U'] = p' \tag{28}$$

Subtracting (28) from (27) and substituting values for ϕ_t, ϕ'_t, ϕ_x, ϕ'_x, we get at the interface, y=0

$$(\rho'-\rho)g\eta + \rho[\frac{(\omega-kU)\omega}{k} - (\omega-kU)U]\eta + \rho'[\frac{(\omega-kU')\omega}{k} - (\omega-kU')U']\eta = p-p'$$

But

$$p-p' = -[\frac{\sigma}{R_{xy}} + \frac{\sigma}{R_{tr}}] \text{ and } R_{xy} = \frac{\partial^2\eta}{\partial x^2} = -k^2\eta \ .$$

Here we have assumed that the effect of transverse surface tension is additive to the effect of axial surface tension. Therefore

$$(\rho-\rho')g - \frac{\rho}{k}[(\omega-kU)^2] - \frac{\rho'}{k}[(\omega-kU')^2] = -k^2\sigma + \frac{\sigma}{R_{tr}\eta} \tag{29}$$

Multiplying throughout by $\frac{1}{k(\rho+\rho')}$ and solving for the velocity, $\frac{\omega}{k} = c$, we get

$$c = \left[\underbrace{\frac{\rho U + \rho' U'}{\rho + \rho'}}_{i} \pm \sqrt{\underbrace{\frac{g}{k}\,\frac{\rho - \rho'}{\rho + \rho'}}_{ii} + \underbrace{\frac{k\sigma}{\rho + \rho'}}_{iii} - \underbrace{\frac{\sigma}{k(\rho + \rho')R_{tr}\eta}}_{iv} - \underbrace{\frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2}}_{v}} \right] \qquad (30)$$

Equation (30) is a very basic result to which we shall often refer. The significance of the terms on the right hand side is:

(i) Mass mean velocity.

(ii) Gravity term acts to smooth out the irregularities when the heavier fluid is below the lighter fluid, and enhances irregularities when the light fluid is below.

(iii) Axial curvature acts to smooth out the irregularities.

(iv) Transverse curvature acts to augment the irregularities.

(v) Inertia term acts to augment the irregularities.

C. Stability of Waves

We are interested in stability of waves of the type

$$\eta = ae^{-ikx}\, e^{ikct}$$

η will grow without bound if c is a negative imaginary number or includes a negative imaginary term. Thus when the term under the radical in equation (30) passes through zero (i.e. goes from positive to negative), k assumes its critical or maximum stable value $k_c = \frac{2\pi}{\lambda_c}$.

Actually the physical system will collapse in the optimum or "most dangerous" wave length, $\lambda_d = \frac{2\pi}{k_d}$. This is the value of λ, for which $|\omega|$ is maximum (i.e. $\frac{d\omega}{dk} = 0$) and for which η grows most rapidly.

1. Taylor's Instability

When two different fluids having a common plane boundary are accelerated in a direction perpendicular to the boundary, any small irregularities in the boundary will tend to change in shape. If the acceleration is directed from the more dense to the less dense medium, the irregularities will tend to smooth out; and if the acceleration is directed from less dense to more dense medium, the irregularities of the interface will tend to grow. This effect is known as Taylor's Instability.

Now from our previous analysis we calculate the critical and most dangerous wave lengths.

Putting $U = U' = 0$ in equation (29) and assuming no curvature in transverse direction [$R_{tr} = \infty$, elementary Taylor's Instability] we get

$$c^2 = \frac{g}{k}\left(\frac{\rho-\rho'}{\rho+\rho'}\right) + \frac{\sigma k}{\rho+\rho'} \tag{31}$$

Here we have taken the heavier liquid as below the lighter liquid <u>i.e.</u> $\rho>\rho'$.

Putting $c^2 = 0$ we get

$$k_c = \sqrt{\frac{g(\rho-\rho')}{\sigma}}$$

or

$$\lambda_c = \frac{2\pi}{\sqrt{\frac{g(\rho-\rho')}{\sigma}}} \tag{32}$$

For $k<k_c$, the R.H.S. of (31) is negative and the interface is unstable.

Example: Calculate critical wave length for water and air interface at normal temperature and pressure:

$$\begin{aligned} \rho &= 1.935 \text{ lbf sec}^2/\text{ft}^4 \\ \rho' &= 0.00234 \text{ lbf sec}^2/\text{ft}^4 \\ \sigma &= 0.005 \text{ lbf/ft} \end{aligned}$$

Therefore

$$\lambda_c = \frac{2\pi}{\sqrt{\dfrac{32.2 \times 1.933}{0.005}}} = 0.057 \text{ ft} = 0.684 \text{ in}$$

Now setting

$$\frac{d\omega}{dk} = \frac{d(ck)}{dk} = 0$$

we get

$$k_d = \frac{1}{\sqrt{3}}\sqrt{\frac{g(\rho-\rho')}{\sigma}}$$

or

$$\lambda_d = \frac{2\pi\sqrt{3}}{\sqrt{\dfrac{g(\rho-\rho')}{\sigma}}} = \sqrt{3}\,\lambda_c \qquad (33)$$

Actually the system will collapse with this most dangerous wave length. For the water and air interface in the preceding example this value is

$$\lambda_d = \underline{1.18 \text{ inches}}$$

2. Helmholtz Instability

This involves the study of instability of the interface when either of the two fluids is moving in a direction perpendicular to the acceleration of the interface, the flow of a gas with velocity U' over water, for example.

Here we discuss either of the two situations:

(i) Surface Tension Neglected

(ii) Gravity is Neglected

(i) Surface Tension Neglected: Equation (30) reduces to the form

$$c = \frac{\rho' U'}{\rho + \rho'} \pm \sqrt{\frac{g}{k}\,\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'\, U'^2}{(\rho + \rho')^2}}$$

Setting the term under the radical sign equal to zero we get

$$k_c = \frac{\rho^2-\rho'^2}{\rho\rho'}\frac{g}{U'^2}$$

$$\lambda_c = \frac{2\pi\rho\rho'}{\rho^2-\rho'^2}\frac{U'^2}{g} \qquad (34)$$

and setting $\frac{d\omega}{dk} = \frac{\partial(ck)}{\partial k} = 0$

we get $k_d = \frac{\rho^2-\rho'^2}{2\rho\rho'}\frac{g}{U'^2}$

or

$$\lambda_d = \frac{2\pi(2\rho\rho')}{\rho^2-\rho'^2}\frac{U'^2}{g} = 2\lambda_c \qquad (35)$$

Example: Obtain the critical and most dangerous wave lengths for air moving over a lake with a speed 30 ft/sec.

$$\lambda_c = \frac{2\pi \times 1.935 \times 0.00234}{(1.935)^2} \times \frac{900}{32.2} = 0.208 \text{ ft}$$

$$\lambda_d = 0.416 \text{ ft} \approx 5 \text{ inches}$$

Thus Helmholtz instability only contributes short wavelength choppiness to the lake's surface.

(ii) Neglecting Gravity: Rewriting equation (30) with $U = 0$, infinite transverse curvature, and zero gravity we get

$$c = \frac{\rho'U'}{\rho+\rho'} \pm\sqrt{\frac{k\sigma}{\rho+\rho'} - \frac{\rho\rho'U'^2}{(\rho+\rho')^2}}$$

Setting the terms under radical sign equal to zero we get

$$k_c = \frac{\rho\rho'}{\rho+\rho'} \frac{U'^2}{\sigma}$$

or

$$\lambda_c = \frac{2\pi(\rho+\rho')}{\rho\rho'} \frac{\sigma}{U'^2} \tag{36}$$

and setting

$$\frac{d\omega}{dk} = 0$$

we get

$$k_d = \frac{2}{3} \frac{\rho\rho'}{(\rho+\rho')} \frac{U'^2}{\sigma}$$

$$\lambda_d = \frac{3\pi(\rho+\rho')}{\rho\rho'} \frac{\sigma}{U'^2} = \frac{3}{2} \lambda_c \tag{37}$$

D. Three Dimensional Cases

All our previous analysis was based on two dimensional waves as we neglected the effect of transverse curvature. Although the transverse curvature effect makes the problem three-dimensional, still a two dimensional analytical model can be established to describe this. The effect of surface tension along the curvature in the transverse direction is same as an additional component of pressure acting to push the interface inwards.

Consider a cylinder of liquid--say a jet moving with a velocity U_g, as shown in Fig. 5. U_g can be set equal to zero without any loss of generality by moving the frame of reference along the jet with the speed of the jet, and we write

$$p-p' = -\sigma \frac{d^2\eta}{dx^2} - \Delta p_{tr} \exp[i(\omega t-kx)]$$

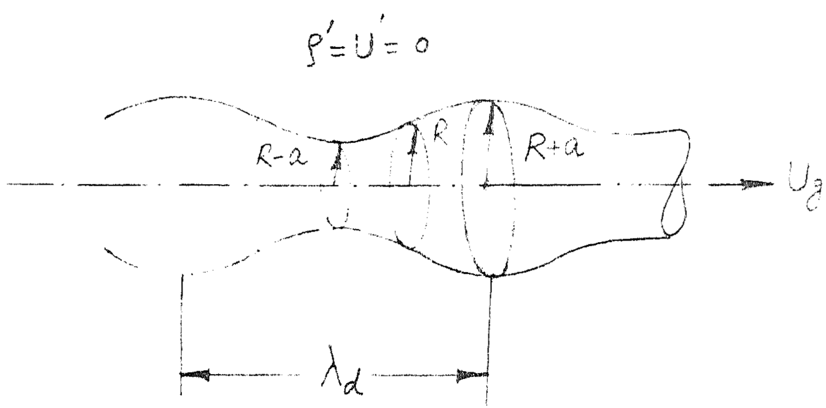


Fig. 5 Jet Moving in a Stationary Fluid of Negligible Density

where a stationary component of transverse pressure has been added to both sides, and

Δp_{tr} = amplitude of transverse pressure oscillation

$\exp[i(\omega t-kx)]$ denotes disturbance effect

The transverse pressure ranges from $\frac{\sigma}{R+a}$ at the peaks to $\frac{\sigma}{R-a}$ at the troughs, therefore it has an amplitude given by

$$\Delta p_{tr} = \frac{1}{2}\left[\frac{\sigma}{R-a} - \frac{\sigma}{R+a}\right] = \frac{1}{2}\left[\frac{R+a-R+a}{R^2-a^2}\right] \approx \frac{\sigma a}{R^2}$$

Substituting all the desired values in equation (30) we get

$$c = \frac{\omega}{k} = \pm\sqrt{\frac{g}{k}\rho + \frac{k\sigma}{\rho} - \frac{\sigma}{k\rho R^2}}$$

But g = 0 in the jet. Therefore we get,

$$c = \pm\sqrt{\frac{k\sigma}{\rho} - \frac{\sigma}{k\rho R^2}}$$

The critical value of k is thus

$$k_c = \frac{1}{R}$$

or

$$\lambda_c = 2\pi R \tag{38}$$

The critical wavelength is equal to the circumference of the jet in this case.

Similarly we can get the most dangerous wave length by putting $\frac{d\omega}{dk} = 0$:

$$k_d = \sqrt{\frac{2}{3}}\Big/R = \frac{0.816}{R}$$

or

$$\lambda_d = \sqrt{6}\ \pi R = 7.7R \tag{39}$$

However, our deep water assumption is not valid here as the jet is not really very deep. In this case $R\ k_d$ is not $>> 1$.

Rayleigh [2] analyzed this problem without using the deep water assumption and got

$$\lambda_c = 2\pi R$$

and

$$\lambda_d = 9.016\,R \tag{40}$$

II. THE PEAK AND MINIMUM BOILING HEAT FLUXES

A. The Model

Figure 6 shows a typical plot of heat flux against liquid superheat. Various boiling regimes are also shown.

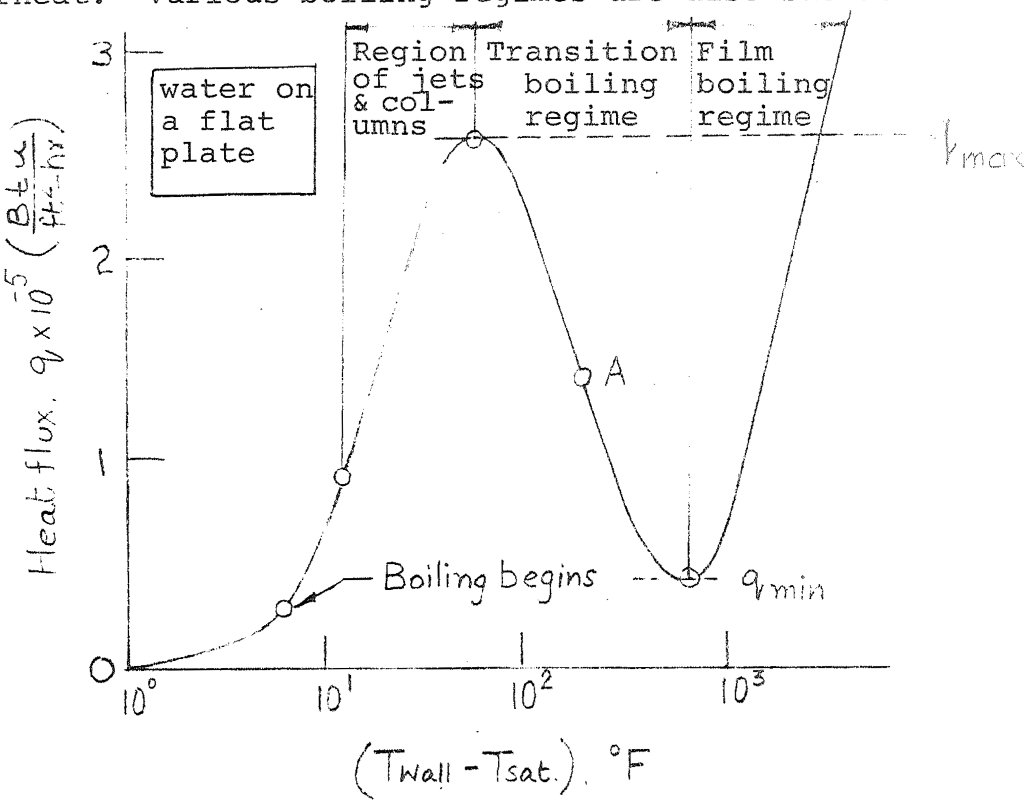


Fig. 6 Typical boiling curve and regimes of boiling

Consider boiling in the transitional boiling region denoted by point A (about 300°F above saturation temperature for water at one atmosphere pressure). The spikes of liquid get close to the surface and evaporate rapidly generating a lot of vapor. The liquid vapor interface is hydrodynamically unstable because the acceleration is directed from the vapor to the liquid. Figs. 7a to 7d show how the liquid vapor interface behaves in the transitional boiling region.

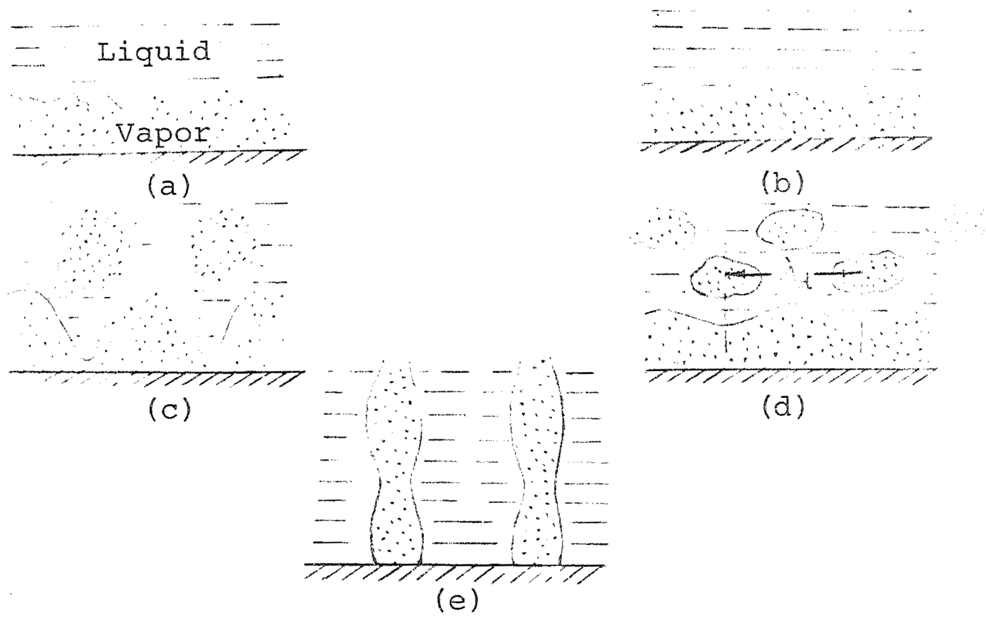


Fig. 7 Typical Interface Shapes in Transitional Boiling

7a The interface has random perturbations distributied over a spectrum of wave lengths.

7b The wavelength near the "most dangerous" one is the first to achieve a finite amplitude. Therefore, as a result of Taylor instability, interface takes a finite geometrical shape.

7c The interface consists of spikes of liquid and of rounded regions similar to cylindrical bubbles which rise into the liquid. In their downward fall, the spikes approach the heater surface and rapidly evaporate, causing an explosive flashing to occur at that point.

7d As the minimum heat flux is approached, the interface stabilizes into a regular Taylor unstable wave form. As a row of bubbles is released, an unstable interface is formed again. The downward flow of liquid, instead of occurring in spikes, will be formed as the trough of the wave. Therefore, successive rows of bubbles above the wave will be displaced by half a wave length.

7e Near the peak heat flux, the rate of evaporation is very high. The release of bubbles appears like vapor explosions. The interface rushes towards the surface, but due to rapid evaporation, it is pushed back violently and vapor is released in the form of explosive jets. The vapor speed at which jets stabilize is given by Helmholtz instability.

The strategies we shall follow in predicting q_{max} and q_{min} -- the limiting heat fluxes at either end of the transition region -- will go as follows:

a. Prediction of Peak Heat Flux (q_{max}):

1. Assume a geometry of vapor outflow (jets).
2. Find U_g for which jets become Helmholtz unstable.
3. Find amount of heat flux, q, required to generate enough vapor to give this U_g in this geometry.

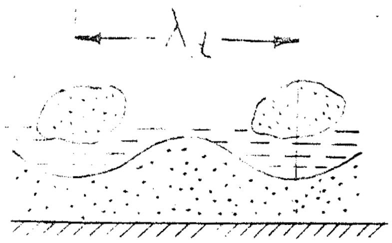
b. Prediction of Minimum Heat Flux (q_{min}):

1. Assume geometry of bubble pattern.
2. Estimate the minimum frequency of wave consistant with the "most dangerous" wave length, λ_d.
3. Find heat flux, q, consistant with vapor outflow required to give this frequency.

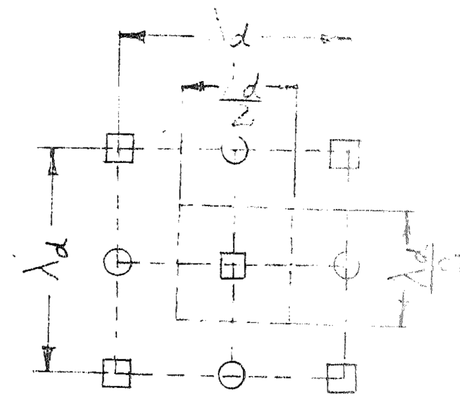
B. Minimum Heat Flux on a Flat Plate

1. The Geometry. Figure 8a shows the shape of interface in the minimum heat flux range. The wave which achieves the finite amplitude is the "most dangerous" one. The bubbles that are forming or leaving the heater surface are spaced a distance λ_d apart. Figure 8b shows top view of the bubble formation on the flat plate heater. Squares show the bubbles that are forming, whereas circles show the bubbles leaving the surface. As the bubbles separate at the nodes, we can safely assume them to be spheres of radius $\frac{\lambda_d}{4}$.

As is seen from Figure 8b, there is one bubble of volume $(\frac{4\pi}{3})(\frac{\lambda_d}{4})^3$ ft^3 per cycle per $(\frac{\lambda_d}{2})^2$ ft^2.



(a)



(b)

Fig. 8 Bubble pattern in minimum heat flux region.

Therefore,

$$\text{volume flux} = f_b \frac{\frac{4\pi}{3\times 64} \lambda_d^3}{1/4\ \lambda_d^2} = f_b \frac{\pi}{12} \lambda_d \frac{ft^3}{ft^2 sec} \tag{41}$$

where f_b = frequency at which bubbles leave heater surface.

2. Frequency. Taking the rate of penetration of the vapor region into the liquid the same as that predicted by Taylor's instability, the relation between the frequency of bubble release f_b, and the rate of penetration of the interface, $\frac{d\eta}{dt}$, then becomes

$$f_b = \frac{1}{\lambda_d} \frac{d\eta}{dt} = \frac{i}{\lambda_d} \omega_{max}\ \eta$$

Lewis [3] observed experimentally that the amplitude, η, of the wave increases at an exponential rate until it reaches the limit of linear wave behavior at $\eta = 0.4\ \lambda_d$. During the exponential growth, the disturbance amplitude, η, increases from an infinitesimal value to approximately 0.4 λ_d. Zuber [4] averaged $\frac{d\eta}{dt}$ over this value and he got,

$$\overline{\frac{d\eta}{dt}} = \frac{1}{0.4\lambda_d}\int_0^{0.4\lambda_d} \frac{d\eta}{dt}\, d\eta = \frac{1}{0.4\lambda_d}\int_0^{0.4\lambda_d} i\omega_{max}\eta\, d\eta$$

$$= 0.2i\ \omega_{max}\ \lambda_d$$

Therefore,

$$f_b = 0.2i\ \omega_{max}$$

Later on Berenson [5] showed that Zuber should have time averaged $\frac{d\eta}{dt}$ rather than averaging over η. Such an average cannot be done because of limited information. He therefore noted that the result would be of the form

$$f_b = ib\omega_{max}$$

where b is determined experimentally.

3. Minimum Heat Flux (q_{min}).

$$q_{min} = \left(\frac{\text{Latent Heat Transport}}{\text{Unit Volume}}\right)\left(\frac{\text{Volume of Vapor}}{\text{Area x Time}}\right)$$

$$= \rho_g\ h_{fg}\ \ 0.2\ i\ \omega_{max}\ \frac{\pi}{12}\ \lambda_d \tag{43}$$

From Equation (33)

$$\lambda_d = \frac{2\pi\sqrt{3}}{\sqrt{\frac{g(\rho_f-\rho_g)}{\sigma}}}$$

and from Equation (31)

$$\omega_{max} = -i\sqrt{k_d}\sqrt{\frac{g(\rho_f-\rho_g)}{\rho_f+\rho_g} - \frac{\sigma k_d^{\ 2}}{\rho_f+\rho_g}}$$

Hence

$$q_{min} = 0.2 \, \rho_g h_{fg} \frac{\pi}{12} \frac{2\pi \sqrt[4]{3}}{\sqrt[4]{\frac{g(\rho_f-\rho_g)}{\sigma}}} \sqrt{\frac{2}{3} \frac{(\rho_f-\rho_g) g}{\rho_f+\rho_g}}$$

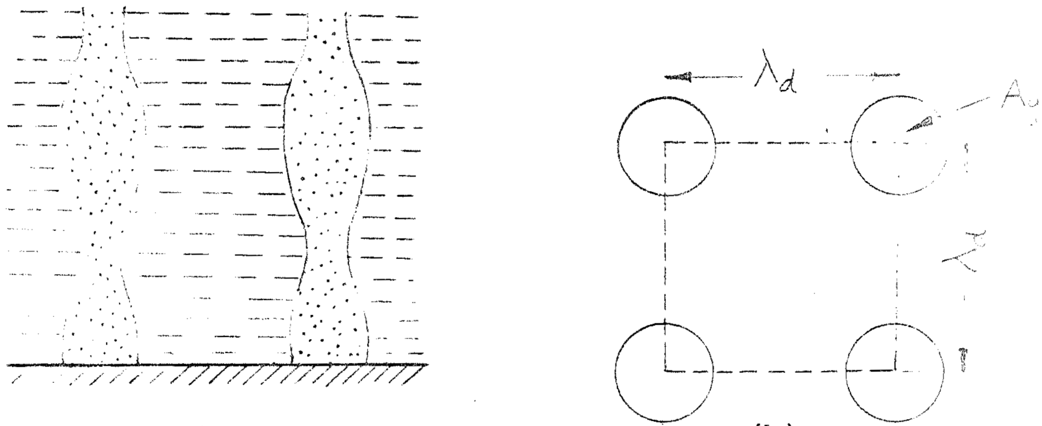
$$= \rho_g h_{fg} \frac{\pi^2}{30} \sqrt[4]{\frac{4}{3}} \sqrt[4]{\frac{\sigma g(\rho_f-\rho_g)}{(\rho_f+\rho_g)^2}}$$

$$= 0.354 \, \rho_g h_{fg} \sqrt[4]{\frac{\sigma g(\rho_f-\rho_g)}{(\rho_f+\rho_g)^2}} \qquad (44)$$

Zuber got a value of q_{min} which is half of this value. The basic difference lies in the fact that he assumed two bubbles were released in an area, λ_d^2, whereas we claim that four are released. The matter is not important, though , since Berenson shows that the experimental constant must be 0.09 instead of 0.354.

C. Peak Heat Fluxes on a Flat Plate

1. Geometry. Figure 9a shows the form of the jets generated from the heater surface at peak heat fluxes. Figure 9b shows the top view of such jets on a flat plate heater. The spacing of the jets is given by Taylor's instability and the wave length at which the jets become unstable would be the shortest one that would be Rayleigh unstable. Helmholtz instability will give the critical speed of such jets.



(a) (b)

Fig. 9 Spacing of jets in peak heat flux region.

There is one jet in an area λ_d^2, therefore,

$$\text{Area of outflow of vapor} = \pi\left(\frac{\lambda_d}{4}\right)^2 = \frac{\pi}{16}\lambda_d^2$$

$$\text{Area of inflow of liquid} = \left(1 - \frac{\pi}{16}\right)\lambda_d^2$$

Hence, by continuity (ρAU = constant)

$$\rho_f U_f \lambda_d^2\left(1 - \frac{\pi}{16}\right) = \rho_g U_g \lambda_d^2 \frac{\pi}{16}$$

or

$$\frac{U_f}{U_g} = \frac{\rho_g}{\rho_f}\frac{\frac{\pi}{16}}{1-\frac{\pi}{16}} \tag{45}$$

2. Critical Velocity of Jets. The shortest wavelength at which jets become Rayleigh unstable is given by equation (40) as

$$\lambda_H = 2\pi R$$

$$= 2\pi\frac{\lambda_d}{4} = \frac{\pi\lambda_d}{2}$$

or

$$k_H = \frac{4}{\lambda_d}$$

But λ_d from equation (33) is

$$\lambda_d = \frac{2\pi \sqrt{3}}{\sqrt{\frac{g(\rho_f-\rho_g)}{\sigma}}}$$

therefore,

$$k_H = \frac{2\sqrt{g(\rho_f-\rho_g)}}{\pi\sqrt{3} \quad \sqrt{\sigma}} \tag{46}$$

Now substituting the critical wave number from equation (46) into equation (30), we get the critical velocity U_{gc} of the jet considering wave propagation velocity to be zero;

$$U_{gc} = \frac{\sqrt{2}}{\sqrt{\pi} \sqrt[4]{3}} \sqrt[4]{\frac{\sigma g(\rho_f-\rho_g)}{\rho_g^{\,2}}} \left[\sqrt{1 + \frac{\rho_g}{\rho_f}} \left(\frac{1}{1 + \frac{\pi}{16-\pi} \frac{\rho_g}{\rho_f}} \right) \right] \tag{47}$$

where the term in brackets is ≈ 1 at low pressures and ≈ 1.13 at critical pressure.

3. Maximum Heat Flux.

$$q_{max} = \rho_g h_{fg} \frac{A_g}{A} U_{gc}$$

where

A_g = area of the jet = $\frac{\pi}{16} \lambda_d^{\,2}$

A = area of the heater subtended by one jet = $\lambda_d^{\,2}$

therefore,

$$q_{max} = \frac{2\sqrt{\pi}}{16 \sqrt[4]{3}} \rho_g h_{fg} \sqrt[4]{\frac{\sigma g(\rho_f-\rho_g)}{\rho_g^{\,2}}}$$

$$q_{max} = 0.119\ \rho_g^{1/2} h_{fg}\ \sqrt[4]{\sigma g(\rho_f - \rho_g)} \qquad (48)$$

Borishanski [6] obtained an experimental expression for q_{max}:

$$q_{max} = 0.13\ \rho_g^{1/2} h_{fg} \sqrt[4]{\sigma g(\rho_f - \rho_g)}$$

D. Cylindrical Geometry

To predict the peak and minimum heat flux on a cylindrical heater, the effect of finite transverse curvature has to be taken into account. Figure 10 shows film boiling near the minimum heat flux on a horizontal cylindrical heater as assumed by Lienhard and Wong [7]. The vapor blanket surrounding the heater is assumed to be sufficiently thin that the smallest radius of interface is negligibly larger than the radius, R, of the heater.

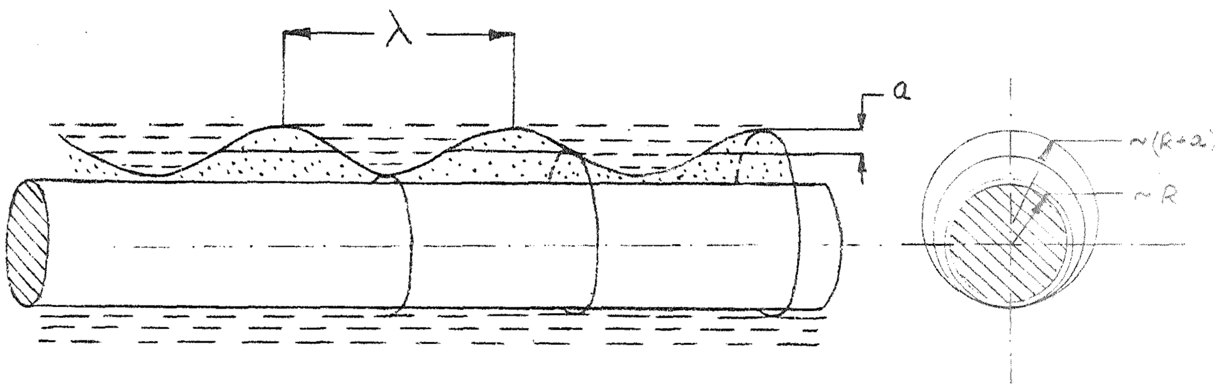


Fig. 10 Interface geometry of minimum heat flux on a horizontal cylindrical heater.

A simple two dimensional model can be used to find the "most dangerous" and critical wave lengths with the effect of transverse curvature treated as an additional pressure difference component, Δp_{tr}

$$\Delta p_{tr} = \frac{\sigma}{2R^2}\,\eta \qquad (49)$$

The volume flux of vapor is

$$\text{vapor flux} = f_b \frac{2}{2\pi R\lambda_d} \frac{4\pi}{3}\left(\frac{\lambda_d}{4}\right)^3$$

$$= f_b \frac{\lambda_d^2}{48R}$$

And the bubble frequency is $f_b = b\, i\omega_{max}$ as in the case of a flat plate . Therefore, if we follow Zuber for the moment and set $b = 0.2$

$$q_{min} = \rho_g h_{fg}\, 0.2\, i\omega_{max} \frac{\lambda_d^2}{48R}$$

From equation (30) after substitution of equation (49) and setting $\frac{d\omega}{dk} = 0$, we get

$$\lambda_d = \frac{2\pi\sqrt{3}}{\sqrt{\frac{g(\rho_f-\rho_g)}{\sigma} + \frac{1}{2R^2}}}$$

Thus

$$\omega_{max} = -i\sqrt{kd}\sqrt{\frac{g(\rho_f-\rho_g)}{\rho_f+\rho_g} - \frac{\sigma k_d^2}{\rho_f+\rho_g} + \frac{\sigma}{(\rho_f+\rho_g)2R^2}}$$

and

$$q_{min} = \frac{\pi^2}{60}\sqrt[4]{3}\left[2g\frac{\rho_f-\rho_g}{\rho_f+\rho_g} + \frac{\sigma}{(\rho_f+\rho_g)R^2}\right]^{1/2}\left[\frac{g(\rho_f-\rho_g)}{\sigma} + \frac{1}{2R^2}\right]^{-3/4}\frac{\rho_g h_{fg}}{R} \qquad (50)$$

Equation (50) represents existing experimental results well if we use $b = .0525$ instead of 0.2. The result will be to replace $\pi^2\, \sqrt[4]{3}/60$ with 0.057 (see [7]).

The reader interested in additional work on both q_{max} and q_{min} on cylinders should consult references [8] and [9].

APPENDIX A

Nomenclature

A	surface area of the heater
A_g	area of vapor jet
a	amplitude of the wave
c	wave velocity
f_b	frequency of bubble release from heater surface
g	acceleration due to gravity
h	depth of the fluid
h_{fg}	latent heat of evaporization
k	wave number $(\frac{2\pi}{\lambda})$
k_c	critical wave number
k_d	"most dangerous" wave number
k_H	Helmholtz critical wave number
p	pressure
q	heat flux
q_{max}	peak heat flux
q_{min}	minimum heat flux
R	radius of cylindrical heater
R_{xy}	radius of curvature in x-y plane
R_{tr}	radius of curvature in transverse direction
t	time
U	free stream fluid velocity
U_g	velocity of vapor jet
U_{gc}	critical velocity of vapor jet
u	velocity component in x direction
v	velocity component in y direction
W	complex potential

z complex number, $x + iy$

Greek Letters

Δp_{tr}	amplitude of transverse pressure oscillations
η	wave height above mean level
λ	wave length
λ_c	critical wave length
λ_d	"most dangerous" wave length
λ_H	Helmholtz critical wave length
ρ	density of fluid
ρ_f, ρ_g	density of saturated liquid and gas, respectively
σ	surface tension
ϕ	potential function
ψ	stream function
ω	frequency

Superscript

prime denotes upper fluid properties

APPENDIX B

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